ARITHMETICAL PROPERTIES OF GEGENBAUER POLYNOMIALS AND SMALL DENOMINATORS ON THE SPHERE

(OPEN PROBLEM)

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Let $\{C_j^\lambda(t) : j = 0, 1, \ldots\}$ be a family of Gegenbauer polynomials. Suppose that λ and $t = \cos\theta$ are fixed, and J is the set of all j satisfying $C_j^\lambda(t) = 0$.

Question 1. *How many j's, and which j's exactly, belong to J?*

The problem is intimately connected with the injectivity and boundedness of the spherical section transform

$$S_\theta f(x) = \int_{x\cdot y=\cos\theta} f(y)d\sigma(y), \qquad x \in S^n, \tag{1}$$

on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, $n \geq 2$ (see [1] for details). Specifically, if $\lambda = (n-1)/2$, the number N of the elements of J determines the dimension $\dim\ker S_\theta$. Denote $\beta = \theta/\pi$ and assume $\beta \neq 1/2$ (the case $\beta = 1/2$ is more or less trivial). For β rational, something is known, see [1]. Namely, in this case $N < \infty$ for $n > 2$ and $N = \infty$ for $n = 2$. The same holds for $\dim\ker S_\theta$. An upper bound for N was evaluated.

If β is irrational, then $N = 0$ for $n = 3$, and nothing is known for $n \neq 3$.

Question 2. *Suppose that β is irrational and the operator S_θ is injective. What can one say about the boundedness of the inverse operator S_θ^{-1}, e.g., in the scale of Sobolev spaces $\mathcal{H}^s$ or L^p_α on S^n?*

The similar question was asked by R.S. Strichartz (in Summer 1997) for the spherical cap transform which integrates $f(y)$ over the geodesic ball $\{x \cdot y \geq \cos\theta\}$. The case of rational β was investigated in [1]. In the irrational case the answer depends on arithmetical properties of β and leads to small denominators for spherical harmonic expansions. The relevant results for $n = 1$ in the framework of harmonic analysis on the circle can be found in the papers by V.I. Arnold, J.-C. Yoccoz, M.R. Herman, Y. Katznelson and others (see [1] for these references).

The following partial results for $n = 3$ were obtained in [1]. We denote by $\mathcal{L}(X, Y)$ the set of linear bounded operators from X to Y, where X and Y are Sobolev spaces under consideration.

THEOREM . *If $n = 3$ and β is irrational, then S_θ is an injective mapping from $\mathcal{H}^s$ into $\mathcal{H}^{s+1}$. In this case the following statements hold:*

(a) If $0 \le \mu < 1$, then $(S_\theta)^{-1} \notin \mathcal{L}(\mathcal{H}^{s+1+\mu}, \mathcal{H}^s)$ for all β.
(b) $(S_\theta)^{-1} \notin \mathcal{L}(\mathcal{H}^{s+2}, \mathcal{H}^s)$ for almost all β.
(c) If $\mu > 1$, then

$$(S_\theta)^{-1} \in \mathcal{L}(\mathcal{H}^{s+1+\mu}, \mathcal{H}^s) \tag{2}$$

for almost all β. In particular, (2) holds for all real algebraic numbers β of degree $d \ge 2$. Moreover, the Hausdorff dimension of the set of all β, for which (2) fails, does not exceed $2/(\mu + 1)$.

(d) If β is a Liouville number (rapidly approximated by rationals), then the boundedness $(S_\theta)^{-1} : \mathcal{H}^s \to \mathcal{H}^r$ fails for any real s, r.

These statements can be proven using the classical tools of diophantine approximation. For $n \ne 3$ the same method leads to a very complicated system of diophantine inequalities and new ideas seem to be needed.

Similar problems can be posed for the Jacobi polynomials and the associated Legendre functions. Apart from the integral geometry, the results can be applied to studying ill-posed Cauchy problems (with shifted data) for singular PDE on S^n [1].

References

[1] B. R u b i n, Generalized Minkowski-Funk transforms and small denominators on the sphere. *Fract. Calc. & Appl. Anal.* **3**, No 2 (2000), 177-203.

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